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Journal of Applied Mathematics and Mechanics 71 (2007) 676–689
Journal of APPLIED MATHEMATICS AND MECHANICS
www.elsevier.com/locate/jappmathmech

Rapid cylindrically and spherically symmetric strong compression of a perfect gas☆

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Received 17 May 2006

Abstract

The problem of the rapid cylindrically and spherically symmetric strong compression of a perfect (non-viscous and non-heat-conducting) gas is solved. The term "rapid" denotes that the compression time is much less than the run time of a sound wave across the initial cylindrical or spherical volume, while the term "strong" in this case means the simultaneous attainment of as large a density and temperature as desired. By definition, rapid compression must begin in a strong shock wave, which propagates to the axis or centre of symmetry. When the shock wave approaches the centre of symmetry this flow is described by the self-similar Guderley equation with an unbounded rise in temperature, pressure and velocity and a finite increase in the density at the centre of symmetry both behind the arriving and behind the reflected shock waves. To obtain as high an increase in the density as desired one must add on a centred compression wave with focus at the centre of symmetry to the overtaking shock wave at the instant it arrives at the centre of symmetry C^--characteristic. Outside a small neighbourhood of the focus one can calculate, by the method of characteristics, the centred wave and the trajectory of the piston which produces it. As for any centred wave, this calculation must be carried out from the centre of symmetry. Since some of the parameters at the focus (certainly the pressure, temperature and velocity of the gas) are unbounded, it is necessary to preface the calculation by the method of characteristics by constructing an analytic solution which holds in a small neighbourhood of the centre of symmetry. Below, after constructing the required solution, the centred waves corresponding to it and the trajectories of the piston producing them are calculated.

Interest in problems of spherically and cylindrically symmetric transient compression has been stimulated by innumerable applications, including plans to realize inertial controlled thermonuclear fusion.[1–6] To achieve this the material of the thermonuclear target must be compressed by a factor of $(0.5–4) \times 10^3$, its temperature must be increased to $(0.3–1) \times 10^8$ K and one must try to satisfy the "ignition condition"

$$I \equiv \int_0^{r_f} \rho dr \geq I_*$$

Here I_* is a known constant, ρ is the density, r is the distance from the centre of symmetry and r_f is the value of r at the boundary of the compressed sphere or cylinder. It is necessary to satisfy these conditions with comparatively low energy costs, while to reduce the effect of viscosity, heat conduction and other dissipative effects the compression and heating up must occur as rapidly as possible. Thus, if ν_* is the kinematic viscosity of the gas while l_* and τ_*

☆ *Prikl. Mat. Mekh.* Vol. 71, No. 5, pp. 744–760, 2007.
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0021-8928/$ – see front matter
doi:10.1016/j.jappmathmech.2007.11.001

are the characteristic length and time of the process, the effect of viscosity will be less the smaller the ratio $\nu_*\tau_*/l_*^2$. Consequently, a reduction in τ_* and a reduction in the viscosity produce the same effect.

The main result of the solution[7] of the variational problem of the energetically optimum compression to a specified ratio c of the initial and final volumes is the detection of the possibility of a relatively rapid shock-free (isentropic) compression "from rest to rest" in a time of the order of the run time t_0 of a sound wave from the boundary of the initial volume to the centre of symmetry.

The energetically optimum isentropic compression from rest to rest enables the density ratio $\rho_f/\rho_0 = c$ and ignition criterion $I \geq I_*$ necessary for controlled inertial thermonuclear fusion to be achieved at a low level of the temperature T_f of the compressed material. In fact, for isentropic compression of a perfect gas $T_f/T_0 = (\rho_f/\rho_0)^{\gamma-1}$, where γ is the adiabatic index, and if $\rho_f/\rho_0 = 10^4$, then $T_f/T_0 = 39.8$ and $T_f/T_0 = 464$ respectively for $\gamma = 7/5$ and $\gamma = 5/3$. When $T_0 = 300$ K this gives a temperature many orders of magnitude less than the required $T_f = (0.3–1) \times 10^8$ K.

For $t_f \ll t_0$ compression to $c \gg 1$ must begin with a shock wave travelling towards the centre of symmetry. In the region of the centre of symmetry the flow is described by the Guderley solution.[8–10] According to this, for an order of magnitude more rapid compression with a strong head shock wave SI the gas density increases by a finite factor. It reaches its maximum value behind the reflected shock wave, where, in the spherically symmetric case, $\rho = 32.3\ \rho_0$ for $\gamma = 5/3$ and $\rho = 145\ \rho_0$ for $\gamma = 7/5$. Behind the reflected shock wave SR the gas moves from the centre of symmetry. The velocity and density of the gas at the centre of symmetry then becomes zero. Whereas zero velocity is a natural consequence of the symmetry of the flow, the fact that the density vanishes is not so obvious. The latter, however, can be explained fairly simply. The point is that the instant when the shock wave SI arrives at the centre of symmetry for a finite density, the pressure p, the temperature and the entropy s behind the SI become infinite. At the centre of symmetry these parameters are also infinite behind the reflected shock wave. Unlike the entropy conserved in the particle, the pressure when a transition occurs from SR to the centre of symmetry (in the rt plane – on the t axis) becomes finite. As a result, as also in the problem of a strong point explosion, in the perfect gas approximation at the centre of symmetry the temperature remains infinite while the density $\rho \sim p/T$ becomes zero.

As already stated, it is possible to obtain densities greater than those behind the reflected shock wave if we add to the particular C^--characteristic of the Guderley solution a fan of characteristics of the same family with focus at the centre of symmetry. In Ref. 11 an attempt was made to solve the singularity which arises at the centre of symmetry, but this was unsuccessful. Below we explain the reason for this lack of success, we obtain the required analytic solution with extremely interesting properties, and then, from it, by using the method of characteristics, we construct the trajectories of the piston, which provide the conditions required for controlled inertial thermonuclear fusion. In the next section, which precedes a discussion of these results, we sum up more than ten years of discussion on the possibility of the isentropic compression of a perfect gas, bounded by a cylindrical or spherical envelope ("piston") from rest to rest.

1. The isentropic compression of a perfect gas

In the problem of the transient compression of a perfect gas at rest with uniform thermodynamic properties at the initial instant of time $t = 0$, from energy considerations it is best to represent the compression after a specified time t_f to a specified mean density $\rho_f = c\rho_0$, where ρ_0 is the initial density, and the degree of compression $c > 1$, with minimum work of the piston. The variational problem of the shock-free one-dimensional compression under the above conditions was formulated for the first time by Sidorov.[12,13] When $t_f \leq t_0$ an exact solution was constructed for the compression of a plane layer of a perfect gas.[12,13] This is illustrated in Fig. 1,a in which x is a Cartesian coordinate, the thick curves are the trajectories of a fixed wall: $x \equiv 0$ and of the piston: $x = x(t)$, while the thin inclined straight lines are the C^--characteristics, focusing at the instant when the compression is completed (when $t = t_f$). When $t_f = t_0$ the characteristics are focused on the fixed wall. If $t_f \leq t_0$, the plane isentropic flow is a simple wave. For this the pressure p is a known function of the gas velocity, while on the piston $dx/dt = 1/t'$ where $t' = dx/dt$. Thus, $p = p(t')$, and the work of compression of the gas is equal to

$$A = -\int_{x_i}^{x_f} p(t')dx \tag{1.1}$$

The solution of Euler's equation of the variational problem with functional (1.1) is elementary: $t' = \text{const} = t_f/(x_i - x_f)$, which corresponds to the motion of a piston in a gas with constant non-zero velocity. This solution, however, does not

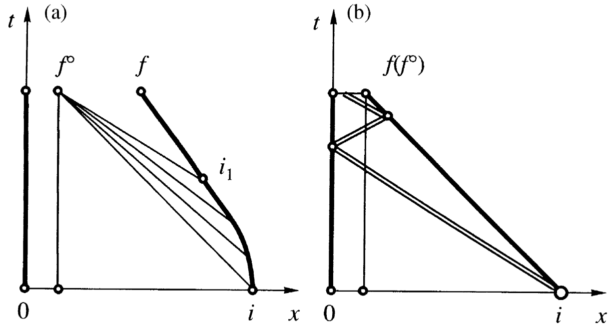

Fig. 1.

satisfy the requirement of shock-free flow, which occurs in the formulation of the problem, or the form of functional (1.1) with known function $p(t')$ obtained as a result of this requirement. For this reason the trajectory of the piston contains an initial part of the boundary extremum ii_1 which does not satisfy Euler's equation. The motion of the piston begins with zero velocity, which increases (in modulus) to the maximum achievable, so that the C^--characteristics only intersect when $t=t_f$. As a result, as is also required from the formulation of the variational problem, when $t<t_f$ the flow is shock-free, while the shock wave, which at once is of finite intensity, only occurs at the instant when the compression is completed (at the point $x = x_{f^\circ} \geq 0, t = t_f \leq t_0$ in the xt plane). When $t_f<t_0$ the shock wave which occurs before reflection from the fixed wall propagates to the left. When $t_f=t_0$ it immediately moves to the right.

For a fixed fairly short time of the process t_f the requirement that it should be isentropic or, which is equivalent, shock-free in general by no means ensures a minimum of the work expended in compression. In fact, suppose that for $t_f<t_0$ the specified coordinate of the piston $x_f = x_{f^\circ} > 0$. In this case the gas is compressed without shock which, prior to compression, was arranged with $x_{f^\circ} \leq x \leq x$. Here the whole of the compressed gas occupies zero volume, i.e. its density and pressure when $t=t_f$ becomes infinite, while the gas close to the fixed wall $(0 \leq x < x_{f^\circ})$ remains uncompressed. It can be shown that for such shock-free compression the work A is infinite. On the other hand, the work required for shock compression to $0 < x_f \leq x_{f^\circ}$ by a piston moving, for example, with constant velocity (Fig. 1,b; the double-line sections are trajectories of the shock wave) is finite. Here the shock compression is energy-wise more advantageous than shock-free compression. Since when $x_f \geq x_{f^\circ}$ the work of shock-free compression is a continuous function of x_f, a similar situation also occurs when there is a finite difference between x_f and x_{f°. Moreover, when there is a shock wave present, the motion of a piston with a constant velocity is not necessarily energetically optimum. Finally, until $x_f < x_{f^\circ}$ shock-free compression is simply impossible.

The variational problem of shock-free compression of a gas by a plane, cylindrical or spherical piston was solved in Ref. 7 in the exact formulation for an arbitrary time t_f. Here, we mean by an exact solution the reduction of the problem to a numerical solution of certain standard problems of the method of characteristics with finite relations on one of the characteristics. The latter are obtained as integrals of the necessary optimality conditions. One of the results of the solution is the discovery of the possibility of isentropic compression from rest to rest. To achieve this the specified compression time t_f must be no less than a certain value t_{f0}. When $t_f \geq t_{f0}$ shock-free compression from rest to rest is energetically certainly more favourable than any other both shock-free and shock. For this reason it is natural to call it ideal compression (IC). According to results obtained in Refs. 14, 15, the ratio $t_{f0}/t_0=2$ when there is no compression $(c=1)$ and $t_{f0}/t_0 \to 1$when $c \to \infty$. Moreover, as was pointed out in Ref. 16, using one-dimensional solutions with any symmetry, ideal compression can also be achieved after a finite comparatively short time using an ideally deformed two-dimensional piston for an arbitrary spatial initial volume, for example a cube. Compression of the spatial volume can be obtained by an infinite number of methods, and they all give a finite volume similar to the initial one.

In ideal compression, the trajectory of a plane cylindrical or spherical piston corresponds to the flow pattern shown in the rt plane in Fig. 2,a. In this figure, in addition to the axes of coordinates and the trajectory of the piston if we show the C^+- and C^--characteristics. We have taken as the scale of length, velocity, time and density here and henceforth the initial coordinate of the piston r_i, the velocity of sound in the uncompressed gas a_0, the ratio r_i/a_0 and the initial density ρ_0, and hence the rectilinear "limiting" C^--characteristic $i0$, which separates the gas at rest (under it) from the moving gas, makes, with the axes of coordinates an angle equal in modulus to 45°. Under the C^+-characteristic c^+f, at least up to $t=t_f$, the gas is also at rest. Its thermodynamic parameters are constant and are determined by the equations

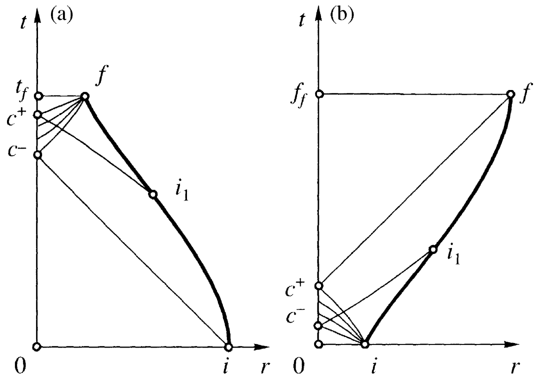

Fig. 2.

of state with equal initial specific entropy $s_f = s_0$ and with a density equal to, for the chosen scales of the specified degree of compression

$$s_f = s_0, \quad \rho_f = c = r_f^{-\nu}, \quad a_f = a(\rho_f, s_f) = a(c, s_0) \tag{1.2}$$

Here the velocity of sound a is a known function of ρ and s, while $\nu = 1$, 2 and 3 in the plane, cylindrical and spherical cases. Similar equalities define all the thermodynamic parameters of the gas at rest in a finite state.

The trajectory of the piston which realises ideal compression, is such that the C^--characteristics which go from its initial part ii_1, being reflected from the t axis as C^+-characteristics, are focused at the point f. At first glance, the possibility of constructing such a trajectory seems problematical. This problem, however, has a simple solution, which is a consequence of the invariance of the equations and the conditions which describe shock-free one-dimensional flow of a perfect gas, with respect to an arbitrary shift in the origin of coordinates of the time and a simultaneous change in the signs of the time and of the r-component of the velocity. After a corresponding shift and the change of signs indicated, the compression problem (Fig. 2,a) becomes an expansion problem (Fig. 2,b) with constant parameters of the gas at rest on the C^--characteristic ic^- known from relations (1.2) and also known parameters (initial parameters in the compression problem) of the gas at rest on the C^+-characteristic c^+f.

The solution of the problem of the expansion of a gas is reduced to successive solution of two standard problems of the method of characteristics. A beam of rarefaction waves ic^-c^+ from the C^--characteristics emerging from the point i, is first calculated. It is calculated up to the point c^+ with known values of the density $\rho = 1$ and the other thermodynamic parameters. The condition for symmetry of the flow on the axis or at the centre of symmetry is that the velocity at the point c^+ should be equal to zero, which is satisfied when calculating the beam ic^-c^+ over the whole section c^-c^+ of the t axis. The trajectory of the advancing piston if is then constructed as the trajectory of a particle starting from the point i, as a result of the solution of a Goursat problem with known C^--characteristic ic^+ and also known rectilinear C^+-characteristic c^+f, obtained from a calculation of a fan of rarefaction waves ic^-c^+, on which the gas velocity is zero, and, according to the choice of the scales, $\rho = a = 1$. The trajectory of the piston and the flow diagram corresponding to the initial problem of compression from rest to rest, are obtained from Fig. 2,b, calculated by the method described, with specular reflection in the r axis, a shift by t_f along the t axis and obvious redesignation of the points.

Fig. 2,b corresponds to the motion of the piston at once with a finite velocity (when $t < 0$ it was at rest), which, strictly speaking, is physically impossible. In contradiction to this, in the case shown in Fig. 2,a there is also no need to instantaneously stop the piston at the instant $t = t_f$. In fact, if when $t > t_f$ we remove the external force applied to the piston, it begins its natural deceleration, and the gas will remain at rest and uniform until the arrival of the shock wave, which moves with finite velocity from the point f to the t axis. At any point with coordinate $0 \leq r < r_f$ the shock wave arrives after a finite time. On the other hand, acceleration from rest to rest is also most simply obtained when the piston accelerates with a finite initial acceleration. In this case the C^--characteristics will not emerge from the point i, but from a known part of the trajectory of the piston next to it, the length of which, like the intensity of the centred wave at the point i, is determined by the condition for obtaining the required density at the point c^+. A transfer from the initial finite velocity of the piston to a finite initial acceleration does not lead to any appreciable complication of the calculation of the total expansion flow from rest to rest. Although in this case the time t_{c+} increases in the compression problem from rest to rest, obtained from the expansion problem, as before $t_{f0}/t_0 \to 1$ when $c \to \infty$. Moreover, now there is no physically unreal instantaneous stop of the piston when $t = t_f$.

It is not surprising if, for physicists unfamiliar with gas dynamics, the possibility of a fairly rapid isentropic multiple expansion or compression of a gas from rest to rest (close to the run time of a sound wave for expansion or across the initial volume for compression) gives rise to amazement (obviously it is assumed to be unnecessary in all courses on thermodynamics to stipulate "infinitely slow" compressions and expansions to ensure these processes to be isentropic).

At the same time, for volumes that include a centre or axis of symmetry, doubts have been raised regarding the possibility of acceleration and deceleration from rest to rest.[17–20] The point is that in the expansion of a cylindrical and spherical piston both with finite and with infinite acceleration, the partial derivatives with respect to time of the density and pressure at the point c^- of the t axis in Fig. 2,b become infinite, to be sure, at "minus infinity", indicating not deceleration but acceleration of the gas. It was asserted in Refs. 17–20 that infinity of the derivatives (irrespective of the sign!) indicates the occurrence of a shock wave. This idea is erroneous in several respects. First, the sign of the infinity is important. In the case of the rarefaction flows corresponding to Fig. 2,b it is negative, which includes the intersecting of similar characteristics in the neighbourhood of the point c^-. Incidentally, the derivatives at the point i, from which the C^--characteristics emerge, separating into a fan and not intersecting, also become "minus infinity". Second, the conversion of the same derivatives into "plus infinity" is insufficient for similar characteristics to intersect and to form a jump. It is precisely this situation that occurs in the neighbourhood of the point c^+ in Fig. 2,a. Third, even the transformation of the same derivatives into "plus infinity" with simultaneous intersection of similar characteristics does not necessarily lead to the occurrence of a shock wave. The simplest example is a centred compression wave with a focal point of similar characteristics on a trajectory of the moving piston with an instantaneous change in its velocity (like at the point f in Fig. 2,a with a stop of the piston at $t = t_f + 0$).

In Refs. 18–20 a "mathematically rigorous" proof (for expanding cylindrical and spherical layers with a fixed internal wall excluding a centre of symmetry) of the possibility of shock-free expansion from rest to rest supposedly presented. The solution of the problems thereby arising (in particular, the Goursat problem) was constructed in the form of series. Without going into detail of their construction, the proof of convergence, etc., we will emphasise that without calculations (although using the same series) one can conclude that it is impossible for the flows described to be shock-free. As an example consider a simple compression wave. For this, the continuous dependence of x on t and on the gas velocity is given by the final formula. However, in this case also, only an additional analysis reveals the intersection of similar characteristics. Only the absence of such intersections serves as a true indicator of the continuity of the flow, which, for hollow volumes, containing a centre of symmetry, rarefaction flows (Fig. 2,b) in the neighbourhood of the point c^- where demonstrated and analysed[16,21], and calculations were also carried out by the method of characteristics in Refs. 22, 23. However, after this, discussions also appeared on the "gradient catastrophe in rarefaction flows" with the introduction of the "rerarefaction effect",[24] which have no physical or mathematical meaning.

An analogue of the problem of the trasient expansion from rest to rest is the stationary problem of the profiling of a nozzle, which accelerates a uniform supersonic flow to a high-velocity uniform flow. In the axisymmetric case in this problem, at the point of intersection of the initial C^--characteristic of the supersonic flow with the axis of symmetry (by analogy with the point c^- in Fig. 2,b) the same feature occurs – corresponding derivatives of the density and pressure become minus infinity. This fact nevertheless, neither in theory nor in numerous calculations, prevents the construction of the required contours of shock-free nozzles. Taking into account the well-known invariance of the equations of the steady flows of a perfect gas, axisymmetric nozzles are converted in a natural way into axisymmetric diffusors, which isentropically slow down the gas. This can be regarded as an additional confirmation of the correctness of the considerations, related to unsteady problems, the diagrams of which are shown in Fig. 2.

A self-similar solution with a shock-free cylindrically and spherically symmetrical unlimited compression was constructed in Refs. 25 and 26. In this solution the total compression time is equal to the run time of a sound wave through the uncompressed gas, but at the instant of cumulation (when $t = 0$) all the parameters, including the density and modulus of the velocity, become infinite. For this reason, and when there is incomplete cumulation, i.e. for a small but non-zero final volume, this compression requires greater energy costs than compression to the same density and temperature from rest to rest. Moreover, the distributions of the temperature, density and velocity of the gas, compressed by this method, are extremely non-uniform with respect to the mass (the Lagrange variable) of the compressed gas. A comparison of the different methods of shock-free compression was carried out in Refs. 11, 27 and 28; in addition to shock-free compression, self-similar compression from rest to rest was included in the comparison with a shock wave travelling from the centre of symmetry of the shut-off gas.[29] For $c \leq 10^6$ the work of this compression is only an amount of the order of 1% greater than that expended in ideal compression.

2. The solution in a small neighbourhood of the focus of a centred wave, adjoining a particular characteristic of the Guderley problem

As was noted above, rapid compression after a time $t_f \ll t_0$ should initiate a strong shock wave. Then, to obtain a greater increase in the density that follows from the Guderley solutions, the trajectory of the piston under the point where a particular C$^-$-characteristic of this solution C_0^- emerges from it, which overtakes the shock wave SI at the instant it arrives at the centre of symmetry, it must change so as to ensure focusing of the fan of C$^-$-characteristics at the same point. For a plane piston such a possibility is elementary. The *xt*-diagram corresponding to it is shown in Fig. 3,a. By increasing the initial velocity of the piston and the velocity of the shock wave SI, one can obtain as small a compression time as desired for a ratio of the densities on the shock wave equal to $(\gamma+1)/(\gamma-1)$. The density as large as desired ensures a centred compression wave $i_1 0 i_2$, which overtakes the shock wave at the point where it is reflected from the plane of symmetry $x=0$.

Suppose, as in Fig. 3, $\tau=-t$, the time t is measured from the instant of arrival of the shock wave SI and the particular C$^-$-characteristic C_0^- at the centre of symmetry, $u=-v$, and v is the gas velocity. If the focus of the C$^-$-characteristics at the origin of coordinates behind the shock wave (SI in Fig. 3,b) when $\nu=2$ and 3 is possible, then their construction must emerge from the centre of symmetry: $r=\tau=0$. When investigating the structure of the flow in the region of the centre of symmetry we will take into account the fact that the flow behind the shock wave SI and of the characteristic C_0^- arriving at the centre of symmetry is non-isentropic with known entropy function $p/\rho^\gamma = S(m)$. Here the Lagrangian variable m, which is constant on the trajectories of the particles, is introduced by the equality

$$dm = r^{\nu-1}\rho(dr - u\,d\tau) = \frac{\rho a}{u+a}dz; \quad z = \frac{r^\nu}{\nu} \tag{2.1}$$

in which the second expression for dm holds on the C$^-$-characteristics.

On the characteristic C_0^-, which coincides with the line of constancy of the self-similar variable of the Guderley solution $\xi = r/\tau^n$, according to well-known results[7–10] and Eq. (2.1), for an appropriate choice of the scales, we have

$$a = m^\alpha, \quad u = \mu m^\alpha, \quad \rho = 1, \quad p = \frac{1}{\gamma}m^{2\alpha}, \quad z = (1+\mu)m, \quad \alpha = \frac{n-1}{\nu n} \tag{2.2}$$

The constant μ and the exponent n or α occurring here are known, and $n<1$. Consequently, $\alpha<0$, and when $m>0$ the density is constant, while a, u and p increase without limit. From relations (2.2) and the conditions for p/ρ^γ in the compression wave to remain a function of m, we obtain that in this wave

$$\rho = a^{1/\kappa}m^\omega, \quad p = \frac{1}{\gamma}a^{\gamma/\kappa}m^\omega, \quad \omega = -\frac{\alpha}{\kappa}, \quad \kappa = \frac{\gamma-1}{2} \tag{2.3}$$

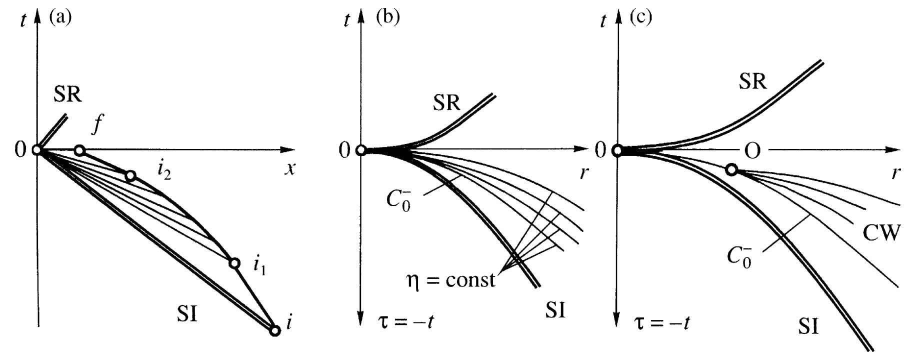

Fig. 3.

If, from the equations of continuity and motion

$$\frac{d\rho}{d\tau}+\rho\frac{\partial u}{\partial r}+(\nu-1)\rho\frac{u}{r} = 0,\quad \frac{du}{d\tau}+\frac{1}{\rho}\frac{\partial p}{\partial r} = 0,\quad \frac{d}{d\tau} = \frac{\partial}{\partial\tau}+u\frac{\partial}{\partial r}$$

using expressions (2.3) to eliminate ρ and p, these equations become

$$\frac{da}{d\tau}+\kappa a\frac{\partial u}{\partial r}+\kappa(\nu-1)a\frac{u}{r} = 0,\quad \kappa\frac{du}{d\tau}+a\frac{\partial a}{\partial r}-\frac{\alpha}{\gamma m}r^{\nu-1}\rho a^2 = 0 \tag{2.4}$$

while the relations equivalent to them, which are satisfied on the $C^{\pm}$-characteristics, take the form

$$\frac{dr}{dm} = \frac{a\mp u}{r^{\nu-1}\rho a},\quad \frac{d\tau}{dm} = \frac{\mp 1}{r^{\nu-1}\rho a},\quad \kappa du\mp da+\left(\kappa\frac{\nu-1}{\nu z\rho}u\pm\frac{\alpha a}{\gamma m}\right)dm = 0 \tag{2.5}$$

The upper (lower) signs correspond to the C^+ (C^-)-characteristics.

Formulae (2.3) predetermine the use of m as one of the independent variables. Taking as the second independent variable the "characteristic" variable η, which is constant on each C^-characteristic of the fan, we reduce system (2.4) to the form (the subscripts denote differentiation with respect to m and with respect to η)

$$\begin{gathered} z_m = \frac{u+a}{\rho a},\quad u_m+\frac{1}{\kappa}a_m-\frac{\alpha a}{\gamma\kappa m}+\frac{\nu-1}{\nu z\rho}u = 0,\quad \rho a z_\eta a_m+(\kappa u_\eta-a_\eta)u-\frac{\alpha\rho a^2}{\gamma m}z_\eta = 0 \\ \tau_m = \frac{(\nu z)^{(1-\nu)/\nu}}{\rho a} \end{gathered} \tag{2.6}$$

Eq. (2.6) hold, in particular, on the C_0^--characteristic. Taking this and Eq. (2.2) into account, from the second equation of (2.6), to determine the coefficient μ, we arrive at a quadratic equation, the roots of which are

$$\mu_{1,2} = \frac{\gamma(\nu n-1)-2(1-n)\pm\sqrt{[\gamma(\nu n-1)-2(1-n)]^2-8\gamma(1-n)^2}}{2\gamma(1-n)} \tag{2.7}$$

To choose the required root, the solution of the Guderley problem, as earlier in Refs. 8–10, is sought in the form

$$a = n\frac{r}{\tau}C_a(\xi),\quad u = n\frac{r}{\tau}C_u(\xi),\quad \rho = \rho_0 C_\rho(\xi),\quad \xi = \frac{r}{\tau^n}$$

where, unlike the previous case, we take as the scale of the velocity and density the ratio nr/τ and the gas density ρ_0 in front of the shock wave SI. It can be shown that the values of the functions C_a and C_u behind SI (with subscript s) and on the particular characteristic of the Guderley problem (with subscript 1 or 2) are given by the formulae

$$C_{as} = \frac{\sqrt{2\gamma(\gamma-1)}}{\gamma+1},\quad C_{us} = \frac{2}{\gamma+1},\quad C_{a1,2} = \frac{1}{1+\mu_{1,2}},\quad C_{u1,2} = 1-C_{a1,2} = \frac{\mu_{1,2}}{1+\mu_{1,2}} \tag{2.8}$$

The values of C_a and C_u obtained form these formulae with $\mu_{1,2}$ from Eq. (2.7), together with the self-similarity index n, found when solving the Guderley problem, are presented in the upper part of the table. In all the versions, apart from the third ($\nu=2$, $\gamma=1.4$), these values of n do not differ from those obtained earlier in Ref. 9. It is possible that the difference in the third version ($n=0.83522$ given in Ref. 9 instead of $n=0.83532$) is simply a misprint.

The part of the integral curve in the C_u, $(C_a)^2$ plane, which gives the solution of the Guderley problem, is the seperatrix of the singular point. In formulae (2.8) the subscripts 1 and 2 label possible values of C_u and C_a at it. The requirement that the integral curve should pass through a point, which in this plane corresponds to the shock wave (the corresponding values have the subscript s), determines the self-similarity index n and the choice between the roots μ_1 and μ_2. In all versions the root μ_1 was the required root.

Earlier[11] the solution of system (2.6) was sought in the form

$$r = \delta(m)\eta,\quad a = \psi(m)A(\eta),\quad u = \psi(m)U(\eta) \tag{2.9}$$

with the required functions $A(\eta)$, $U(\eta)$, $\delta(m)$ and $\psi(m)$, where $\delta(0)=0$ while $\psi(0)=\infty$. This solution was constructed for all versions presented in the upper part of the table. In the first five of these the required functions with $\delta(m)$ and $\psi(m)$, which satisfy the requirements formulated above, were obtained. In the sixth version they were complex. However, in the first five cases also the C^--characteristics of the centred wave obtained was not situated above the particular characteristic of the Guderley problem but under it, i.e. these solutions turned out to be physically meaningless. On the other hand, in the problem of cylindrical and spherical compression waves, adjacent to "uniform rest", the solution in the form (2.9) was identical with the known solution of the problem of unlimited cumulation.[25,26] The fact that, in the latter case, representations (2.9) gave a correct solution, while for the problem with a bow shock wave SI it is physically meaningless, is a paradox, without explaining which one cannot count on success.

When analysing the reasons for the occurrence of this paradox we note a fundamental difference between these problems. On the initial characteristic of the wave adjacent to uniform rest, we have $a=1$, $u=0$, i.e. there are no singularities, and "perturbations" a and u, defined by formulae (2.9), which increase without limit, are the main ones. On the other hand, in the Guderley problem a and u on the particular characteristic increase without limit as $m \to 0$. Taking this into account in this problem the quantities z, a and u can be represented in the form

$$z = (1+\mu)m + \varphi(m)Z(\eta), \quad a = m^{\alpha} + \psi(m)A(\eta), \quad u = \mu m^{\alpha} + \psi(m)U(\eta) \tag{2.10}$$

where the first terms, which are identical with the distributions (2.2) on the C_0^--characteristic, are much greater as $m \to 0$ than the second terms, i.e.

$$(1+\mu)m \gg \varphi(m)Z(\eta), \quad m^{\alpha} \gg \psi(m)A(\eta), \quad \mu m^{\alpha} \gg \psi(m)U(\eta) \tag{2.11}$$

In view of this and relations (2.10), the expression for the density (the first equality of (2.3)) in a small neighbourhood of the centre of symmetry takes the form

$$\rho = 1 + \frac{\psi A}{\kappa m^{\alpha}} \tag{2.12}$$

Substituting expressions (2.10) into the first equation of system (2.6), taking relations (2.11) and (2.12) into account and also the fact that the principal terms in Eq. (2.10) satisfy the same equation, we obtain

$$Z = \frac{\psi}{m^{\alpha}\dot{\varphi}}\Omega = c_0\Omega, \quad \Omega = \frac{\kappa U - [1+(1+\kappa)\mu]A}{\kappa}; \quad \psi = c_0 m^{\alpha}\dot{\varphi} \tag{2.13}$$

with unknown constant c_0. These equations are consequences of the fact that Z, A, U and Ω are functions of η, while ψ and φ are functions of m. Here and henceforth a dot denotes differentiation with respect to m, while a prime denotes differentiation with respect to η.

Similarly, from the second equation of system (2.6), after eliminating Z using Eq. (2.13), we obtain

$$(\kappa U + A)c_2 + \frac{\kappa\omega}{\gamma}A + \frac{\kappa(\nu-1)\mu}{\nu(\mu+1)}\left(\frac{U}{\mu} - \frac{\Omega}{c_1} - \frac{A}{\kappa}\right) = 0, \quad \frac{\ddot{\varphi}}{\dot{\varphi}} = \frac{c_1}{m}, \quad \frac{\dot{\psi}}{\psi} = \frac{c_2}{m} \tag{2.14}$$

Integrating the second and third of these equations, we obtain

$$\varphi = C_{\varphi}m^{c_1}, \quad \psi = C_{\psi}m^{c_2} = C_{\varphi}c_0c_1m^{\alpha-1+c_1}$$

Equating the constants C_{φ} and C_{ψ} to unity, in accordance with the definition of the functions φ and ψ, we obtain

$$\varphi = m^{1+\Delta}, \quad \psi = m^{\alpha+\Delta}, \quad \Delta = c_1 - 1, \quad c_0 = 1/c_1, \quad c_2 = \alpha + c_1 - 1 \tag{2.15}$$

The left-hand side of the first equation of (2.14) is a linear homogeneous function of A and U with constant coefficients, and hence for the functions $A(\eta)$ and $U(\eta)$, which are not identically equal to zero, we have

$$U(\eta) = kA(\eta) \tag{2.16}$$

with a constant k, like c_1, which is to be determined. Taking equality (2.16) and the expression for c_2 (the last equality of (2.15)) into account the first equation of (2.14), relating the required constant, can be written in the form

$$(1+\kappa k)(\alpha - 1 + c_1)c_1 + \frac{\kappa\omega}{\gamma}c_1 + \frac{(\nu-1)\mu}{\nu(\mu+1)}\left[\frac{\kappa k - \mu}{\mu}c_1 + \frac{(1+\kappa)\mu + 1 - \kappa k}{\mu+1}\right] = 0 \tag{2.17}$$

The second equation for determining c_1 and k are obtained after substituting expressions (2.10) into the third equation of system (2.6). By expressing the derivative U′ and Z′ in terms of A′ using the first equation of (2.13) and equality (2.16), we obtain

$$\{2(1-n)[(1+\kappa)\mu + 1 - \kappa k] + (\kappa k - 1)\nu n\gamma c_1\mu\}A' = 0$$

For an identically unequal to zero derivative A′ the factor in front of it is equal to zero. From this we obtain

$$c_1 = 2(1-n)\frac{(1+\kappa)\mu + 1 - \kappa k}{(1-\kappa k)\nu n\gamma\mu} \tag{2.18}$$

When $c_1 = c_{10} = 0$, Eqs. (2.17) and (2.18) are identical and give

$$k = k_0 = \frac{(1+\kappa)\mu + 1}{\kappa} \tag{2.19}$$

Hence, $c_{10} = 0$ and k_0 is one of the solutions of this system, where k_0 is one of the roots of the cubic equation obtained after eliminating c_1 from Eqs. (2.17) and (2.18). Knowing the root (2.19) we can reduce the cubic equation to the following quadratic equation

$$\begin{aligned} &bk^2 - 2dk + f = 0 \\ &b = \kappa^2\{\gamma^2 n^2(\nu-1)\nu\mu^3 + 2(1-n)[\gamma(1+n\nu-n)\mu^3 + (\gamma n\nu - \gamma n + 2n + 4\kappa)\mu^2 + \\ &+ \gamma\mu + 2(n-1)(2\mu+1)]\} \\ &d = \kappa\gamma[(n\nu y - n - 1)n(\nu-1)\mu^2 - (1-n)^2(\mu+1)^2 - (1-n)n(\nu-1)(2\mu+1)]\mu \\ &f = [\gamma n(\nu-1) - 2 + 2n][2n\kappa + \gamma n(\nu-1) - 2 + 3n]\mu^3 + \\ &+ 2(1-n)\{2(1-n)(3\mu^2 + 3\mu + 1) - \gamma n[(3\mu+1)\nu - \mu]\mu\} \end{aligned} \tag{2.20}$$

For $\mu = \mu_1$ the roots k_1 and k_2 of Eq. (2.20) together with k_0, corresponding to them, according to equality (2.18), the quantities c_{11} and c_{12} and the quantities $\Delta_1 = c_{11} - 1$ and $\Delta_2 = c_{12} - 1$ (we recall that for all versions $c_{10} = 0$ and, consequently, $\Delta_0 = -1$), the exponent α and $1/\rho_0$ – the density on the particular C^--characteristic, refered to the density in front of the shock wave, are collected in the lower part of the table. According to the data obtained and formulae (2.15), for the functions φ and ψ inequalities (2.11) are only satisfied for positive Δ, i.e. when $\Delta = \Delta_1$. Below Δ_1, μ_1 and k_1 are written without a subscript.

In all cases $\Delta \ll 1$. In the table we show values of $10^{-3\nu\Delta}$, representing the smallness of the factor Δ. If, on the C_0^--characteristic $m = 1$ when $r = 1$, then $m = 10^{-3\nu}$ when $r = 10^{-3}$, therefore the values of $10^{-3\nu\Delta}$ show that even at such small distances from the centre of symmetry the corrections related to the centred compression wave and equal to zero at the centre of symmetry, are quantities of the order of unity.

We will take as the characteristic variable the function A, putting $\eta = A(\eta)$. By definition, the quantity η, like the function $A(\eta)$, is equal to zero on the initial C^--characteristic of the fan of compression waves (the Guderley solutions on the particular C^--characteristic) and increases with distance form it. Finally, in a small vicinity of the centre of symmetry we will have

$$\begin{aligned} &z = (\mu+1)m(1 + C_z\eta m^\Delta), \quad C_z = \frac{\kappa k - 1 - (1+\kappa)\mu}{\kappa(\mu+1)(1+\Delta)} \\ &a = m^\alpha(1 + \eta m^\Delta), \quad u = m^\alpha(\mu + k\eta m^\Delta), \quad \eta \ge 0 \end{aligned} \tag{2.21}$$

The similar representation for τ, obtained from the last equation of system (2.6), has the form

$$\tau = \frac{\nu^{1/\nu}n(1 + C_\tau\eta m^\Delta)}{(\mu+1)^{(\nu-1)/\nu}}m^{1/(\nu n)}, \quad C_\tau = -\left(\frac{\gamma+1}{\gamma-1} + \frac{\nu-1}{\nu}C_z\right)\frac{1}{1+\nu n\Delta} \tag{2.22}$$

The coefficients C_z and C_τ are given in the last rows of the table. The values k_0, k_2 and Δ_2 given in them will not henceforth be necessary.

3. Features of the calculation of the compression wave

According to formulae (2.21) and (2.22), the corrections to the Guderley solution due to the centred compression wave are proportional to ηm^Δ and, due to the fact that the exponent Δ is positive, they vanish at the centre of symmetry. Theoretically they are also small in a certain "small neighbourhood" of the centre of symmetry, confirming, at first glance, that the hypothesis that the solution of the problem of the collapse of a cylindrical or spherical shock wave is independent (self-similar), in this neighbourhood, of the law of the piston motion, is correct. Indeed, for similar hypotheses to be correct, law of the piston motion must satisfy the "common position" condition. The latter excludes any "special adjustments", and precisely the adjustment (the choice of the trajectory of the piston which ensures

Table 1

Version number	1	2	3	4	5	6
ν	2	3	2	3	2	3
γ	5/3	5/3	1.4	1.4	1.2	1.2
$n\times 10^5$	81563	68838	83532	71717	86116	75714
$C_{as}\times 10^4$	5590	5590	4410	4410	3149	3149
$C_{us}\times 10^4$	7500	7500	8333	8333	9091	9091
$\mu=\mu_1$	1.302	1.282	1.887	1.885	2.976	3.016
$C_{a1}\times 10^4$	4344	4382	3464	3466	2515	2490
$C_{u1}\times 10^4$	5657	5618	6536	6534	7485	7510
μ_2	0.921	0.936	0.757	0.758	0.560	0.553
$C_{a2}\times 10^4$	5205	5165	5691	5689	6410	6441
$C_{u2}\times 10^4$	4795	4835	4309	4311	3590	3559
$-\alpha\times 10^4$	1130	1509	986	1315	806	1069
$1/\rho_0$	4.75	5.19	7.57	8.49	15.0	17.3
k_0	8.21	8.13	16.3	16.3	42.7	43.2
$k=k_1$	2.397	2.161	4.098	3.768	8.480	7.968
$\Delta\times 10^5=\Delta_1\times 10^5$	437	481	1103	1401	1761	2374
$10^{-3\nu\Delta}$	0.941	0.905	0.859	0.748	0.784	0.784
k_2	0.26	0.24	1.04	1.06	3.51	3.55
$-\Delta_2$	0.70	0.60	0.71	0.62	0.73	0.64
$-C_z$	2.513	2.602	4.188	4.287	8.466	8.564
$-C_\tau$	2.724	2.243	3.835	3.050	6.568	5.020

focusing of the C^--characteristics at the centre of symmetry) occurs in this problem. Since, however, $\Delta \ll 1$, the region of self-similarity has practically zero dimensions. This is obviously also a manifestation of the effect of the above-mentioned adjustment.

To construct intense compression waves, the characteristic variable η, which defines the perturbation of the velocity of sound, must increase to a value necessarily exceeding unity. Because of the smallness of Δ this leads to the fact that the second terms in expressions (2.10), despite the assumption that inequalities (2.11) are satisfied, which was used when obtaining formulae (2.21) and (2.22), become predominant at extremely small values of m. For example, when $\eta = 10$ for the first version (see the Table 1) calculations using the formulae obtained are only valid up to $m < 10^{-400}$.

In order not to have to carry out calculations for such small and, correspondingly extremely large, values of $m^\alpha > 10^{40}$, we will make a replacement of variables. We will take as the new variables

$$\lambda = \ln m,\ \zeta(\lambda,\eta) = z/m,\ A(\lambda,\mu) = m^{-\alpha}a,\ U(\lambda,\eta) = m^{-\alpha}u,\ \rho = \rho(\lambda,\eta) = A^{1/\kappa} \tag{3.1}$$

with the previous characteristic variable and density and with the quantities A and U having a quite different meaning than previously. This should not lead to any misunderstandings, since A and U from Section 2 are not used any further. The last relation of (3.1) is a consequence of the definition of the new A and the first formula of (2.3).

For fixed $\lambda = \lambda_0$, where, in view of what was stated above, λ_0 is a comparatively large negative number (of the order of 10^2–10^3), in the neighbourhood of the centre of symmetry the functions ζ, A, U and τ (with subscript 0), according to relations (3.1), (2.21) and (2.22) are given by the formulae

$$\begin{gathered}\zeta_0 = (\mu+1)(1+C_zH),\quad A_0 = 1+H,\quad U_0 = \mu + kH\\ \tau_0 = \nu^{1/\nu}n\frac{1+C_\tau H}{(\mu+1)^{(\nu-1)/\nu}}\exp\frac{\lambda_0}{\nu n};\quad H = \eta\exp(\Delta\lambda_0),\quad \eta_C \ge \eta \ge 0\end{gathered} \tag{3.2}$$

The constant η_C defines the intensity of the compression wave. As follows from the definition of λ and from formulae (3.2), for large negative λ_0 for all $\eta_C \ge \eta > 0$ the non-zero m_0, $z_0 = \zeta m_0$, r_0 and τ_0 can be neglected. In contrast to this, due to the smallness of Δ the differences of ζ_0, A_0, U_0 and ρ_0 from their values on the initial C^--characteristic

$$\zeta_0^- = \mu+1,\quad A_0^- = 1,\quad U_0^- = \mu,\quad \rho_0^- = 1,\quad \Phi_0^- = \Phi(\lambda,0),\quad \lambda_0 \le \lambda \le 0 \tag{3.3}$$

will be quantities of the order of unity. In order that these differences, nevertheless, should not violate inequalities (2.11), the quantity λ_0 must be assumed to be equal to

$$\lambda_0 = \frac{1}{\Delta}\ln\frac{\varepsilon}{\eta_C K}, \quad K = \max\left(1, \frac{k}{\mu}, |C_z|, |C_\tau|\right), \quad 0 < \varepsilon \ll 1 \tag{3.4}$$

In the variables (3.1) the equations of the characteristics (2.5), used in the method of characteristics to calculate the compression wave when $\lambda_0 \le \lambda \le 0$ and $0 \le \eta \le \eta_C$, are replaced by

$$d\zeta = \left(\frac{A \mp U}{A^{(1+\kappa)/\kappa}} - \zeta\right)d\lambda, \quad \kappa dU \mp dA + \left[\frac{(\nu-1)\kappa U}{\nu\zeta A^{1/\kappa}} + \alpha\frac{\gamma\kappa U \pm (1-\gamma)A}{\gamma}\right]d\lambda = 0 \tag{3.5}$$

where, as in relations (2.5), the upper (lower) signs correspond to the C^+ (C^-)-characteristics.

In addition to differential relations (3.5), to determine τ along the C^--characteristics, we integrate the differential equation

$$d\tau = \frac{d\lambda}{(\nu\zeta)^{(\nu-1)/\nu}A^{(1+\kappa)/\kappa}}\exp\frac{\lambda}{\nu n} = \frac{dM}{(\nu\zeta)^{(\nu-1)/\nu}A^{(1+\kappa)/\kappa}}; \quad M = \nu n\exp\frac{\lambda}{\nu n} \tag{3.6}$$

which is equivalent to the second equation of (2.5). As calculations have shown, the use of the variable M instead of λ when integrating Eq. (3.6) ensures far higher accuracy.

The calculation by the method of characteristics is carried out in the direction in which η and λ increase: from the trajectory $\lambda = \lambda_0$ with a value of λ_0 calculated from formula (3.4), and with the functions ζ_0, A_0, U_0 and τ_0, defined by formulae (3.2), and from the initial C^--characteristic, ζ, A and U on which are constant, the values of which are given by formulae (3.3). When calculating each new point one initially finds λ, ζ, A and U by integration of Eq. (3.5). After this, the time τ is determined by integrating Eq. (3.6) along the C^--characteristic, and the other variables are calculated using the final formulae

$$z = \zeta\exp\lambda, \; r = (\nu z)^{1/\nu}, \; m = \exp\lambda, \; a = m^\alpha A, \; u = m^\alpha u, \; \rho = A^{1/\kappa}, \; p = \rho\gamma^{-1}a^2$$

4. The results of calculations

The solution obtained and the method of characteristics with successive calculation of the C^--characteristics, corresponding to increasing values of η, enable us to construct the flow to any fixed trajectory of the particles – the lines m – const, for example, to m = 1. It can be taken as the required trajectory of the piston.

The two versions of the C^--characteristic and the trajectory of the piston, obtained as a result of the calculations, are shown in Fig. 4 for $\nu = 3$ and $\gamma = 5/3$ (a) and $\gamma = 6/5$ (b). The coordinate r and the time t relate to r_p and τ_p, where p is the point of intersection of the trajectory of the piston (the upper curve) with the C_0^--characteristic (the lower curve). In these examples, along the trajectory of the piston $1 \le \rho \le 10^4$. For this method of scaling r and t, the versions with $\nu = 2$ for the same γ hardly differ from those presented in Fig. 4.

Graphs of ρ against τ^o, where $\tau^\circ = \tau/\tau_p$, along the trajectories of the piston, obtained for the six versions presented in the table, are given on a logarithmic scale in Fig. 5. Since $\rho = 1$ on the C_0^--characteristic, all the curves 1–6 (on a

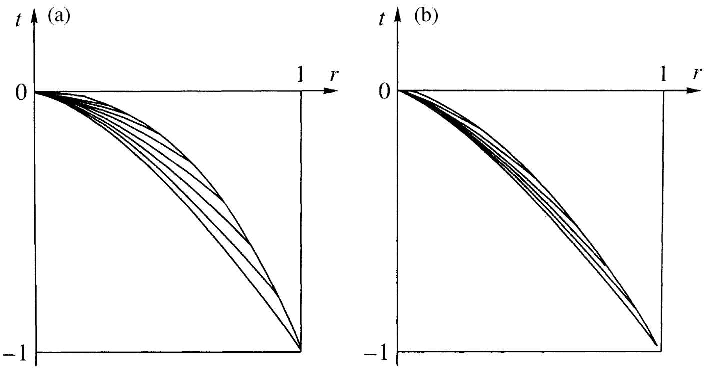

Fig. 4.

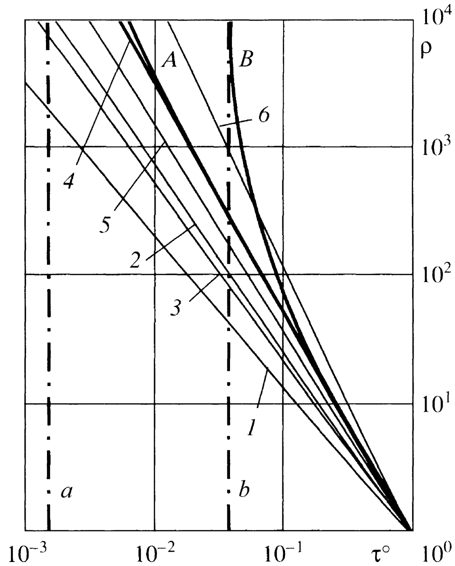

Fig. 5.

logarithmic scale close to straight lines) in Fig. 5 depart from the origin of coordinates. Multiplication of the quantities ρ, given in Fig. 5, by $1/\rho_0$ from the lower part of the table gives the ratio of the density to its value in front of the shock wave SI.

After constructing the solution with a beam of compression waves focusing at the centre of symmetry, it is natural to consider the solution when they are focused at a point situated on the same particular characteristic of the Guderley problem, but not at the centre of symmetry, and for $m_0>0$. The diagram of this flow is shown in Fig. 3,c (CW is a compression wave and O is its focal point), and the corresponding $\nu=3$, $\gamma=7/5$ and the two values of $m_0=10^{-6}$ and 10^{-3} of the dependence of ρ on τ^o along the trajectories of the piston give curves A and B in Fig. 5.

When $m_0>0$ the flow at the focal point is described by the formulae for a plane centred simple wave (the conditions for constancy at this point of the right Riemann invariant and entropy), i.e. at the focal point for constant $m=m_0$, z, r and τ, the variables U and A from the preceding section or the initial velocity u (with a “minus” sign) and the velocity of sound a satisfy the linear relations

$$\kappa U - A = \kappa\mu - 1, \quad \kappa u - a = m_0^{\alpha}(\kappa\mu - 1) \tag{4.1}$$

with the previous formulae for the density and pressure: $\rho=A^{1/\kappa}$, $p=\rho a^2/\gamma$.

Relations (4.1), at least outwardly, have nothing in common with the solution constructed above. Despite this, curves A and B in Fig. 5 only depart from curve 4 as they approach the focal points with $m_0>0$, which is easily explained since for $m_0>0$ the focal point of the C$^-$-characteristics correspond to small but nevertheless finite τ_0° as the density increases without limit. For these values of $\tau_0^\circ=\tau_0/\tau_p>0$ curves A and B have vertical asymptotes (a and b respectively). On the C_0^- characteristic, on which the focal point differing from the centre of symmetry lie, $\tau^\circ=m^{1/(\nu n)}$, and hence the coordinate of the asymptotic $\tau^\circ=m_0^{1/\nu n}$.

After the solution with focusing of the C$^-$-characteristics at the centre of symmetry is constructed, the mutual closeness of the curves A and B as one moves away from the corresponding asymptote is also represented in a fairly natural way. In fact, the solution constructed in Section 2 can be considered as the limit of the solution with a focal point for $m_0>0$, as $m_0\to 0$. The fundamental difficulty of this passage to the limit is not eliminated due to the presence of the asymptote. Nevertheless, if this passage to the limit is possible, the corresponding distributions and curves must be close as one moves away from the asymptote.

What is more astonishing and even paradoxical is the correctness of the solution obtained in the calculations outside a small neighbourhood of the centre of symmetry. It turned out that formulae (2.21) for a and u or (3.2) (without the subscript 0) for A and U

$$\begin{aligned} a &= m^{\alpha}(1+\eta m^{\Delta}), \quad u = m^{\alpha}(\mu + k\eta m^{\Delta}) \\ A &= 1+\eta\exp(\Delta\lambda), \quad U = \mu + k\eta\exp(\Delta\lambda), \quad \eta \geq 0 \end{aligned} \tag{4.2}$$

in the sense described below hold for all $\exp\lambda_0 \leq m \leq 1$ or $\lambda_0 \leq \lambda \leq 0$ with λ_0 defined by formula (3.4). Namely, the substitution of the expressions for a and u or A and U (4.2), defined by the formulae into the equations for z_m and τ_m, which hold on the C^{-}-characteristics (for $\eta = \text{const}$) (see formulae (2.6))

$$z_m = \frac{u+a}{\rho a} = \frac{U+A}{\rho A}, \quad \tau_m = \frac{(\nu z)^{(1-\nu)/\nu}}{\rho a} = \frac{(\nu z)^{(1-\nu)/\nu}}{\rho m^{\alpha} A}; \quad \rho = A^{1/\kappa}$$

and their integration from $m = m_0 = \exp\lambda_0$ to $m = 1$ with initial conditions $z_0 = z(m_0, \eta) = (1+\mu)m_0$ and $\tau_0 = 0$ gives the trajectory of the piston and the distributions of the parameters along it, which hardly differ from those obtained by the method of characteristics. A single difference occurs in the value at the points of the trajectory of the characteristic variable η. Since, however, η is an auxiliary variable, this difference is unimportant. The statement in the last two paragraphs holds for all the values of ν and γ.

5. Conclusion

The results presented above have shown that it is possible in principle simultaneously to compress and heat up a gas to the densities and temperatures required to obtain controlled inertial thermonuclear fusion, and, which is extremely important, in a time that is an order of magnitude less than the run time of a sound wave across the uncompressed target. Dissipative effects and the actual thermodynamics, different from the thermodynamics of a perfect gas, inevitably change both formulae and numerical results, but hardly so much that the required conditions become unachievable. The instability of the flow constructed is more important. The instability of the Guderley solution established previously in Ref. 30 manifests itself in the fact that, as the radius to the centre of symmetry of the shock wave decreases to a negligibly small neighbourhood of it (with respect to the Lagrange variable), spatial perturbations destroy the symmetry of the flow. Later (as the reflected wave moves away from the centre of symmetry) outside this neighbourhood the symmetry of the flow is re-established. A similar instability also occurs for centred compression waves. Another matter is the fact that to use for the practical realisation of strong compression it is proposed a layered system with contact discontinuities, which are not present in the gas-dynamic formulation considered above. The presence of contact discontinuities in accelerating flows with a shock wave generates a Rayleigh-Taylor and Richtmeyer-Meshkov instabilities,[31,32] which may manifest themselves in the whole flow.

Theoretically by, the solution constructed is interesting at least as an example of both the conservation (but only in a negligibly small region of the centre of symmetry), and of the change (outside this neighbourhood) of the self-similar Guderley solution. In this connection, it is worth attempting (though this wont be simple) to solve the similar problem with the centred wave replaced by a second shock wave, travelling to the centre of symmetry simultaneously with the first. Finally, the paradoxical fact of the practical applicability of the solution constructed not only in the neighbourhood of the focal point but also over the whole compression wave, requires an explanation.

Acknowledgements

I wish to thank N. I. Tillyayeva for calculations by the method of characteristics, Kh. F. Valiyev for solving the Guderley problem numerically, G. G. Chernyi and V. A. Simonenko for a stimulating assessment of the research and V. A. Belokon’ for useful discussions.

This research was supported financially by the Russian Foundation for Basic Research (05-01-00846) and within the framework of the Program for the Support of the Leading Scientific Schools (NSh-9902.2006.1).

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Translated by R.C.G.